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# Minimum-Energy Control of Two-Link Manipulator with Pure State Constraints

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## **Abstract**

This paper presents an analysis and numerical solutions of the minimum-energy control of two-link robot manipulator. The minimum-energy control point-to-point trajectory is investigated subject to control constraints and state constraints on the angular velocities. The numerical solutions are solved by transforming the original problem into a nonlinear programming problem. The mathematical analysis of the optimal control problems is done based on the numerical results using an indirect method. The necessary conditions can be stated as a multi-point boundary value problems.

**Keywords:** *optimal control, minimum-energy control, direct method, indirect method, robot manipulator*

## **1. Introduction**

Robot manipulators are used for variety tasks in industry. The important performances of the robot manipulators are the speed and energy when its work. Therefore minimum energy point-to-point trajectory of two-link robot manipulators are investigated subject to

control and state constraints. The point-to-point control of multiple link manipulators can be applied to accurate aiming of an industrial robot or a multi-body spacecraft.

The methods for solving optimal control can be classified generally into two main categories: direct and indirect methods. The direct method solve the optimal control problem by discretizing the control and/or the state variables, transforming the optimal control problem into a Nonlinear Programming Problem, NLP, see eg Betts [1], Von Stryk and Bulirsch [2], Seywald and Kumar [3], while the indirect methods are based on the solving of the necessary conditions derived from the Pontryagin Maximum Principles (Pontryagin et al. [4]).

## 2. Optimal Control Problem

The general optimal control problem is to find an admissible control  $\mathbf{u}$  to optimise the performance index in the following general form (Bryson and Ho [5]):

$$\mathcal{J}[\mathbf{u}] = \phi[\mathbf{x}(t_f, t_f)] + \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u}, t) dt \quad (1)$$

subject to the dynamic equations, terminal conditions and boundary conditions

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}, t) \\ \Psi[\mathbf{x}(t_f), t_f] &= 0 \\ \mathbf{x}(t_0) &= 0 \end{aligned}$$

Here  $\mathbf{x}(t) \in \mathbf{R}^n$ ,  $\mathbf{u}(t) \in \mathbf{R}^l$ , and  $\Psi \in \mathbf{R}^k$ . Assume that the functions  $\phi$ ,  $L$ , and  $f$ , respectively, are continuously differentiable with respect to all their arguments. The Hamiltonian function is defined with Lagrange multipliers  $\boldsymbol{\lambda}(t) \in \mathbf{R}^n$  as

$$H = \boldsymbol{\lambda}^T f + L \quad (2)$$

The minimum principle requires that the control  $\mathbf{u}$  minimise  $H$ :

$$\mathbf{u}^* = \arg \min_{\mathbf{u} \in \Omega} H(\mathbf{x}^*, \boldsymbol{\lambda}^*, \mathbf{u}, t) \quad (3)$$

where  $\Omega$  is the set of admissible piecewise continuous control values and  $\mathbf{x}^*$ ,  $\boldsymbol{\lambda}^*$ , and  $\mathbf{u}^*$  are the extremal of the state, costate, and control variables. The state, costate variables

and the Hamiltonian satisfy the following conditions:

$$\dot{\mathbf{x}}^T = H_\lambda \quad (4a)$$

$$\dot{\lambda}^T = -H_x \quad (4b)$$

$$\lambda^T(t_f) = \left[ \frac{\partial \phi}{\partial \mathbf{x}} + \nu^T \frac{\partial \Psi}{\partial \mathbf{x}} \right] \Big|_{t=t_f} \quad (4c)$$

$$H(t_f) = - \left[ \frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \Psi}{\partial t} \right] \Big|_{t=t_f} \quad (4d)$$

$$H_u = 0 \quad (4e)$$

where  $\nu$  is a constant multiplier vector of the dimension of the constraint  $\Psi$ . Control and state inequality constraints are augmented to the Hamiltonian, and additional necessary conditions are obtained as a result. These necessary conditions depending on the type of the state constraint.

### 3. Problem Formulation

We consider two-link manipulator as developed by Wie, Chuang and Sunkel [6]. The dynamic behaviour of the system is described by the following state equations,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{I_2 u_1 - (\alpha I_2 + I_4 \cos x_3) u_2 + I_2 I_4 (x_2 + x_4)^2 \sin x_3 + I_4^2 x_2^2 \sin x_3 \cos x_3}{I_2 I_3 - I_4^2 \cos^2 x_3} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{-(I_2 + I_4 \cos x_3)(u_1 + I_4(x_2 + x_4)^2 \sin x_3) + (\alpha I_2 + I_3 + (1 + \alpha) I_4 \cos x_3) u_2 - I_4(I_3 + I_4 \cos x_3) x_2^2 \sin x_3}{[I_2 I_3 - I_4^2 \cos^2 x_3]} \end{aligned}$$

where  $\alpha$  is a system parameter used to denote the type of torque applied to the second link,  $\alpha = 1$  for a joint torque and  $\alpha = 0$  for a direct torque,  $I_1$  is the mass moment of inertia of the first link with respect to the shoulder axis scaled by  $T_{max}$ ,  $I_2$  is the mass moment of inertia of the second link with respect to the elbow axis scaled by  $T_{max}$ :

$$\begin{aligned} I_3 &= I_1 + m_2 L_1^2 / T_{max} \\ I_4 &= m_2 r L_1^2 / T_{max} \end{aligned}$$

The minimum energy problem is investigated as the performance index

$$\min \mathcal{J}(u) = \int_0^{t_f} \sum_{i=1}^2 (u_i)^2 dt \quad (5)$$

subject to initial conditions  $\mathbf{x}_0$  and final conditions  $\mathbf{x}_{t_f}$ .

The final time  $t_f$  has to be prescribed in order to obtain useful solution. The control  $u_i$  are bounded as follows:

$$|u_i(t)| \leq 1, \quad i = 1, 2 \quad (6)$$

The state variables are constrained by

$$|x_i(t)| \leq x_{i_{max}}, \quad i = 1, \dots, 4 \quad (7)$$

### 3.1. Unconstrained problem

The problem is transformed into Mayer problem by introducing variable  $x_5$ , where

$$\dot{x}_5 = \sum_{i=1}^2 (u_i)^2, \quad x_5(0) = 0, \quad \min \mathcal{J}(u) = x_5(t_f) \quad (8)$$

Thus the Hamiltonian for the unconstrained problem can be defined by

$$H^{free} = \lambda_{x_1} \dot{x}_1 + \lambda_{x_2} \dot{x}_2 + \lambda_{x_3} \dot{x}_3 + \lambda_{x_4} \dot{x}_4 + \lambda_{x_5} \sum_{i=1}^3 (u_i)^2 \quad (9)$$

The costate equations, defined by  $\dot{\lambda}^T = -H_x$

$$\dot{\lambda}_{x_1} = 0 \quad (10)$$

$$\begin{aligned} \dot{\lambda}_{x_2} = & -\lambda_{x_1} - \frac{\lambda_{x_2}(2I_2I_4(x_2 + x_4)\sin x_3 + 2I_4^2x_2\sin x_3\cos x_3)}{I_2I_3 - I_4^2\cos^2 x_3} \\ & - \frac{\lambda_{x_4}(-2I_4x_2\sin x_3(I_3 + I_4\cos x_3))}{I_2I_3 - I_4^2\cos^2 x_3} \\ & - \frac{2I_4(x_2 + x_4)(I_2 + I_4\cos x_3)\sin x_3}{I_2I_3 - I_4^2\cos^2 x_3} \end{aligned} \quad (11)$$

$$\begin{aligned} \dot{\lambda}_{x_3} = & -\frac{1}{I_2I_3 - I_4^2\cos^2 x_3} \\ & \left\{ \lambda_{x_2}(-I_4\sin x_3u_2 + I_2I_4(x_2 + x_4)^2\cos x_3 + I_4^2x_2^2(\cos^2 x_3 - \sin^2 x_3)) \right. \\ & + \lambda_{x_4}(I_4\sin x_3u_1 - (1 + \alpha)I_4\sin x_3u_2 - I_4(I_3 + I_4\cos x_3)x_2^2\cos x_3 \\ & \left. - I_4(I_2 + I_4\cos x_3)(x_2 + x_4)^2\cos x_3 - I_4^2\sin^2 x_3(x_2^2 + (x_2 + x_4)^2) \right\} \\ & - \frac{1}{(I_2I_3 - I_4^2\cos^2 x_3)^2} \left\{ 2I_4^2\cos x_3\sin x_3(\lambda_{x_2}\dot{x}_2 + \lambda_{x_4}\dot{x}_4) \right\} \end{aligned} \quad (12)$$

$$\begin{aligned} \dot{\lambda}_{x_4} = & -\frac{\lambda_{x_2}(2I_2I_4(x_2 + x_4)\sin x_3)}{I_2I_3 - I_4^2\cos^2 x_3} - \lambda_{x_3} \\ & - \frac{\lambda_{x_4}(2I_4(I_2 + I_4\cos x_3)(x_2 + x_4)\sin x_3)}{I_2I_3 - I_4^2\cos^2 x_3} \end{aligned} \quad (13)$$

$$\dot{\lambda}_{x_5} = 0 \quad (14)$$

The controls can be derived explicitly from  $H_{\mathbf{u}} = 0$  as follows:

$$u_1 = \frac{I_2(\lambda_4 - \lambda_2) + I_4\lambda_4 \cos x_3}{2\lambda_5(I_2I_3 - I_4^2 \cos^2 x_3)} \quad (15)$$

$$u_2 = \frac{I_2\alpha(\lambda_2 - \lambda_4) - I_3I_4 + I_4(\lambda_2 - (1 + \alpha)\lambda_4) \cos x_3}{2\lambda_5(I_2I_3 - I_4^2 \cos^2 x_3)} \quad (16)$$

The transversality condition (eq. [4c]) gives

$$\lambda_5(t_f) = \frac{\partial \mathcal{J}}{\partial x_5(t_f)} = 1 \quad (17)$$

The control  $u_i$  are constrained as follows:

$$|u_i(t)| \leq 1, \quad i = 1, 2 \quad (18)$$

The final time  $t_f$  is fixed on 0.8 sec. Figure 1 shows that  $u_1$  is directly on the minimum value. Then follows by unconstrained case (see Eq. [15]). Finally  $u_1$  is saturated on the maximum value. While  $u_2$  is unconstrained along the optimal trajectory.

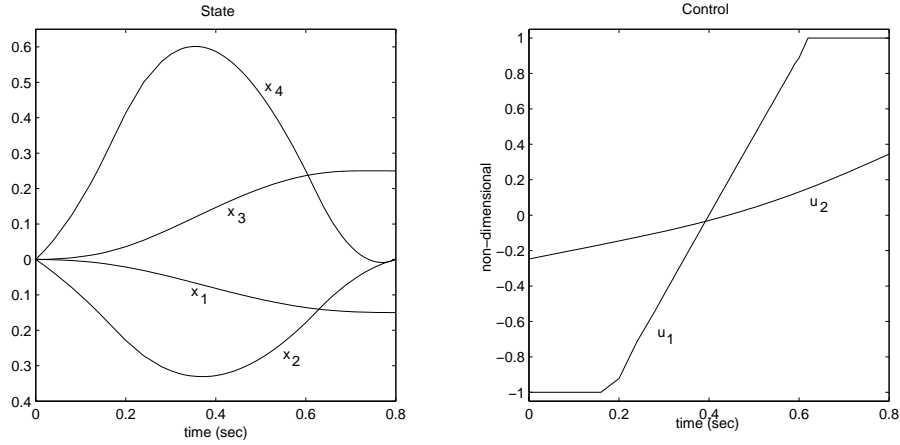


Figure 1: The computational results of the unconstrained case

### 3.2. Constrained problem

The state constraints can be written as

$$S_i := x_i - x_{i_{max}} \leq 0 \quad (19)$$

Consider the following equations:

$$\frac{\partial}{\partial u_i} S_i^{(1)} = 0, \quad \frac{\partial}{\partial u_i} S_i^{(2)} \neq 0, \quad i = 1, 3 \quad (20)$$

and

$$\frac{\partial}{\partial u_i} S_i^{(1)} \neq 0, \quad i = 2, 4 \quad (21)$$

where

$$S_i^{(k)} := \frac{d^k}{dt^k} S_i, \quad i = 1, \dots, 4 \quad k = 1, 2, \dots \quad (22)$$

From eq. [20] and [21] we obtain that the state constraints  $x_1$  and  $x_3$  are second order state constraints and the state constraints  $x_2$  and  $x_4$  are first order state constraints. The Hamiltonian for constrained problem becomes

$$H^{cons} = H^{free} + \eta_1 S_1^{(2)} + \eta_2 S_2^{(1)} + \eta_3 S_3^{(2)} + \eta_4 S_4^{(1)} \quad (23)$$

The costate equations can be derived as in the unconstrained case by considering whether the constraints active or not.

## 4. Numerical Example

This section presents an example for the constrained minimum-energy problem. The initial conditions are  $x_0 = [0, 0, 0, 0]$  and the final conditions are  $x_{t_f} = [-0.15, 0, 0.25, 0]$ . The time  $t_f$  is fixed on 0.8 sec and  $\alpha = 1$ . The state constraints are  $|x_i(t)| \leq 0.4$ ,  $i = 1, \dots, 4$ . The computational results are based on the direct collocation (DIRCOL) by Von Stryk [7].

Figure 2 shows that the state constraint  $x_4$  is active while the other state constraints are not active. When the state constraint  $x_4$  is active the Hamiltonian can be defined by

$$H^{cons} = H^{free} + \eta_4 S_4^{(1)} \quad (24)$$

The control  $u_1$  is saturated directly on the minimum value at the beginning . Then follows by unconstrained case. Finally the control  $u_1$  is saturated on the maximum value. Furthermore the control  $u_2$  is not constrained along the optimal trajectory.

## 5. Conclusions

The optimal trajectory of the minimum-energy of two-link manipulator is presented. The computational results are based on the direct methods. The advantage of the direct methods is that the user does not have to analyse further into the problem by deriving costate variables, jump conditions or switching structures.

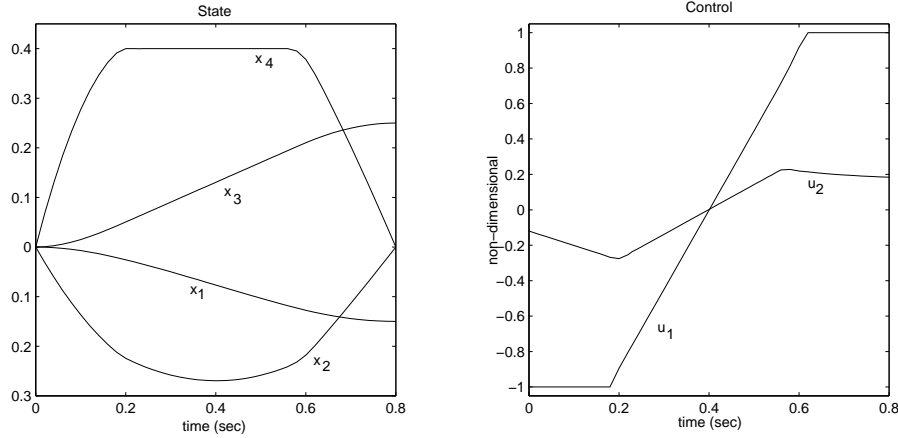


Figure 2: The computational results of the constrained case

The main drawback of the direct methods is that they produce several minima solutions and the solutions are less accurate than the indirect methods. To overcome these problems it is necessary to use the direct method solutions as a starting analysis and initial guesses for the indirect methods.

The main advantage of the indirect methods is that they produce very accurate result. The major difficulties of the indirect methods are that the user must derive the costate variables, jump conditions and switching structures. It is very difficult to define where the jump conditions and switching structures should occur without knowing the direct methods solutions as a priori estimate.

## References

- [1] J. T. Betts. Survey of numerical methods for trajectory optimization. *Journal of Guidance, Control, and Dynamics*, 21(2):193–207, Mar–Apr 1998.
- [2] O. von Stryk and R. Bulirsch. Direct and indirect methods for trajectory optimization. *Annals of Operations Research*, 37:357–373, 1992.
- [3] H. Seywald and R. R. Kumar. Some recent developments in computational optimal control. *IMA Volumes in Mathematics and its Applications*, 93:203–234, 1997.
- [4] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. John Wiley & Sons, New York, 1962.

- [5] A. E. Bryson and Y. C. Ho. *Applied Optimal Control. Optimization, Estimation, and Control. Revised Printing.* Hemisphere Publishing Corporation, New York, 1975.
- [6] B. Wie, C-H. Huang, and J. Sunkel. Minimum-time pointing control of two-link manipulator. *Journal of Guidance, Control, and Dynamics*, 13(5):867–873, 1990.
- [7] O. von Stryk. *User's guide for DIRCOL - a direct collocation method for the numerical solution of optimal control problems.* Technische Universität Darmstad, November 1999.